ALGEBRAIC CURVES
EXERCISE SHEET 5

Unless otherwise specified, k is an algebraically closed field.

Exercise 1.

Let $n \geq 1$ and $I, J \subseteq k[X_0, \ldots, X_n]$ be ideals. For $d \geq 0$ we denote by $k[X_0, \ldots, X_n]_d$ the subspace of forms of degree d and $I_d = I \cap k[X_0, \ldots, X_n]_d$ (resp. $J_d = J \cap k[X_0, \ldots, X_n]_d$). Show that:

(1) If I, J are homogeneous, then $I + J$, IJ and $rad(I)$ are homogeneous.
(2) If I is homogeneous, I is prime if, and only if, for all *homogeneous* $f, g \in k[X_0, \ldots, X_n]$, $fg \in I \Rightarrow f \in I$ or $g \in I$.
(3) I is homogeneous if, and only if, $I = \bigoplus_{d \geq 0} I_d$ (the right-hand side being a direct sum of abelian groups). Give an example of how this fails for non-homogeneous ideals.
(4) If I is homogeneous, then there is a well-defined notion of forms of degree d in $\Gamma = k[X_0, \ldots, X_n]/I$ and the corresponding spaces Γ_d, $d \geq 0$ are finite-dimensional over k.

Exercise 2.

Let $R = k[X, Y, Z]$ and $F \in R$ be an irreducible form of degree $n \geq 1$. Consider $V = V(F) \subseteq \mathbb{P}^2_k$ and $\Gamma = R/(F)$. For $d \geq 0$, we denote by Γ_d the subspace of forms of degree d in Γ (see previous exercise).

(1) Construct an exact sequence $0 \to R \stackrel{\times F}{\to} R \to \Gamma \to 0$, where $\times F$ denotes multiplication by F in R.
(2) Show that, for $d > n$:
$$dim_k(\Gamma_d) = dn - \frac{n(n-3)}{2}$$

Exercise 3.

Let $V = V(Y - X^2, Z - X^3) \subseteq \mathbb{A}_k^3$. Show that:

(1) $I(V) = (Y - X^2, Z - X^3)$.
(2) $ZW - XY \in I(V)^* \subseteq k[X, Y, Z, W]$, but $ZW - XY \notin ((Y - X^2)^*, (Z - X^3)^*)$.

In particular, this shows that, for $F_1, \ldots F_r \in k[X_1, \ldots, X_n]$, the following inclusion can be strict: $(F_1^*, \ldots, F_r^*) \subseteq (F_1, \ldots, F_r)^*$.

Exercise 4.

Let $n \geq 1$ and $T : \mathbb{A}_k^{n+1} \to \mathbb{A}_k^{n+1}$ be a linear change of coordinates (i.e. a linear automorphism of k^{n+1}). As it preserves lines through the origin it induces $T : \mathbb{P}_k^n \to \mathbb{P}_k^n$, what we call a *projective change of coordinates*.

(1) Show, that one can send any $n+1$ points in $\mathbb{P}^n$ not lying on a hyperplane to any other $n+1$ points not lying on a hyperplane via a linear change of coordinates.
(2) Formulate and prove a similar statement for hyperplanes instead of points.

Exercise 5.

Show that any two distinct lines in $\mathbb{P}_k^2$ intersect in one point.

Exercise 6.

Let $m, n \geq 1$ and $N = (n+1)(m+1) - 1 = mn + m + n$. We consider $\mathbb{P}_k^n$ with projective coordinates $X_0, \ldots, X_n$, $\mathbb{P}_k^m$ with projective coordinates $Y_0, \ldots, Y_m$ and $\mathbb{P}_k^N$ with projective coordinates $T_{00}, T_{01}, \ldots, T_{0m}, T_{10}, \ldots, T_{nm}$. We also denote the affine coverings of $\mathbb{P}_k^n$, $\mathbb{P}_k^m$, $\mathbb{P}_k^N$ associated to these coordinates as follows: $U_i = \{X_i \neq 0\}$, $V_j = \{Y_j \neq 0\}$ and $W_{ij} = \{T_{ij} \neq 0\}$.

Define the *Segre embedding* $S : \mathbb{P}_k^n \times \mathbb{P}_k^m \to \mathbb{P}_k^N$ by the formula:

$$S([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = [x_0 y_0 : x_0 y_1 : \ldots : x_n y_m]$$

(1) Show that S is well-defined and injective.
(2) Let $Z = V(T_{ij}T_{kl} - T_{il}T_{kj},\ 0 \leq i, k \leq n,\ 0 \leq j, l \leq m) \subseteq \mathbb{P}_k^N$. Show that $S(\mathbb{P}_k^n \times \mathbb{P}_k^m) = Z$ (more specifically, $S(U_i \times V_j) = Z \cap W_{ij}$ for all i, j).
(3) Show that the topology induced on $\mathbb{P}_k^n \times \mathbb{P}_k^m$ by the Zariski topology of $\mathbb{P}_k^N$ via the Segre embedding is different from the product topology.